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Diffraction of water waves by slender barriers

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Abstract. Linear water-wave theory is used to tackle the problem of diffraction of surface waves by a fixed slender barrier in deep water for two basic situations: (i) when the barrier is partially immersed, and (ii) when the barrier is completely submerged. Analytical expressions for the first-order corrections to the reflection and transmission coefficients are derived in terms of integrals involving the shape functions describing the two sides of the slender barrier. A relatively straightforward perturbation technique is used along with the application of Green's theorem in the fluid region. Corresponding analytical expressions representing the reflection and transmission coefficients are also deduced, (i) for a nearly vertical barrier and (ii) for a vertically symmetric slender barrier, as special cases for both the problems. For a nearly vertical barrier it is observed, analytically, that there is no first-order correction to the transmitted wave at any frequency. Computations for the reflection and transmission coefficients up to $O(\varepsilon)$, where ε is a small nondimensional number, are also performed and presented here.

Key words: diffraction, water waves, submerged barriers, deep water, perturbation method.

1. Introduction

To the author's knowledge, the problem involving the diffraction of water waves by a just immersed vertically symmetric slender barrier was first formulated by Shaw [1]. That author applied a method based on an integral-equation formulation, although the details were omitted. However, the problem of diffraction of water waves by a fixed slender barrier in deep water, for the two basic situations: (i) when the barrier is partially immersed and immersed up to a finite depth a below the free surface, (ii) when the barrier is submerged from a finite depth b below the free surface (FS) and extending infinitely downwards, is considered in the present paper. A method essentially based on a perturbation technique, which is of a type first introduced into this class of boundary-value problems (BVPs) by Mandal and Chakrabarti [2] (henceforth referred to as MC [2]), is employed here to handle the problem under consideration.

In the linearised theory of water waves, diffraction problems involving obstacles admit of exact solutions only for a limited number of cases, for instance, when the obstacles are in the form of thin plane barriers in deep water the motion is then two-dimensional when the barriers are either vertical (*cf*. Dean [3], Ursell [4], Evans [5]) or in the case of a partially immersed barrier inclined at special angles (*cf*. John [6]). Water-wave diffraction problems associated with nearly vertical barriers have become increasingly important and few works have been published on the subject in recent years. The first problem involving a nearly vertical barrier was tackled by Shaw [1]. He considered the diffraction of water waves by a partially immersed curved, but almost vertical barrier. Shaw [1] used Green's theorem (GT) to reduce the problem to the solution of a singular integral equation (SIE), wherein the associated SIE was solved up to first order by a perturbation technique, and corrections up to first order for the reflection and transmission coefficients were obtained there in terms of integrals involving the shape function describing the curved barrier. Later, MC [2] used a simplified perturbation technique and applied this directly to the BVPs, exploiting Evans's [7] idea of the application of GT to determine these corrections for the reflection and transmission coefficients. They considered the problem of diffraction of water waves by a fixed nearly vertical barrier. Subsequently, both mathematical techniques based on an integral equation formulation similar to that of Shaw [1], and a suitable exploitation of Evans's [7] idea along with an appropriately designed perturbation technique used earlier by MC [2], have been utilized successfully by Mandal and Kundu [8] in considering the problem of scattering of water waves by a submerged nearly vertical finite plate. Recently, Vijaya Bharathi and Chakrabarti [9] considered the problem involving a submerged nearly vertical barrier and the complete solution of the first-order BVP was given there (*cf*. [9]) by a method which involves the use of complex-function theory, the Schwartz reflection principle, and reduction to a system of two uncoupled Riemann-Hilbert problems. More recently, the problem of diffraction of water waves by a partially immersed nearly vertical barrier has been considered by Mandal and Banerjea [10]. They obtained the complete solution for the first-order BVP by using Havelock's expansion of the water-wave potential in a manner proposed by Ursell [4].

For the present investigation, a method based on a perturbation analysis (*cf*. MC [2]) and Evans's [7] idea of utilizing GT in the fluid region is used to handle the problem, and the general expressions for the corrections to the reflection and transmission coefficients up to first order are obtained in terms of integrals involving the shape functions describing the two sides of the slender barrier. As a special case, it is verified that, when the two sides of the slender barrier become identical, known results for a nearly vertical barrier given by Shaw [1], as well as MC [2], are recovered. Thus, the problem under consideration may be regarded as the generalisation of the problem of diffraction of water waves by a nearly vertical barrier as considered in [1,2] to that by a slender barrier. Another special case yields the corresponding analytical expressions, representing these corrections to the reflection and transmission coefficients, for a vertically symmetric slender barrier. We have carried out numerical calculations of the reflection and transmission coefficients, up to first order, assuming a particular shape of the slender barrier, symmetric slender barrier and nearly vertical barrier.

2. Formulation

The usual assumptions of linearised water-wave theory ensure the existence of a velocity potential $\Phi(x, y, t)$, and for simple harmonic motions we may write $\Phi = \mathrm{Re}\{\phi(x, y)\exp(-i\sigma t)\}$, where σ is the circular frequency. Thus, the problems under consideration can be mathematically described, in terms of the time-independent velocity potential $\phi(x, y)$, by the following BVPs:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in} \quad y > 0, \tag{1}$$

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{2}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad S, \tag{3}$$

with $r^{1/2}\nabla\phi$ bounded near an edge of the slender barriers and (4)

$$\phi, \nabla\phi \to 0 \quad \text{as} \quad y \to \infty, \tag{5}$$

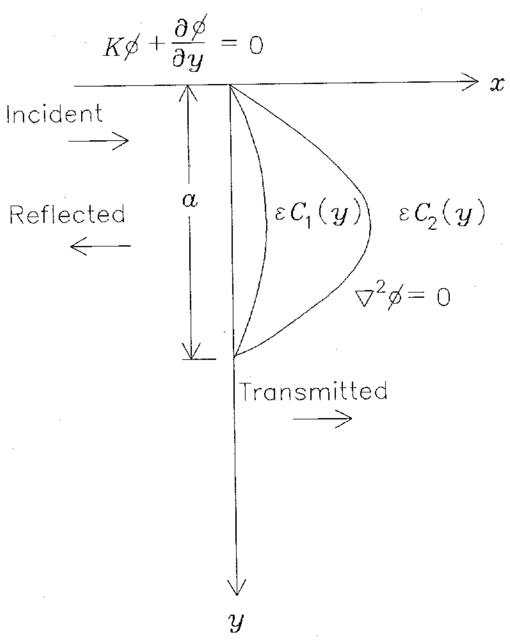

Figure 1a. Slender barrier $(0 < y < a)$.

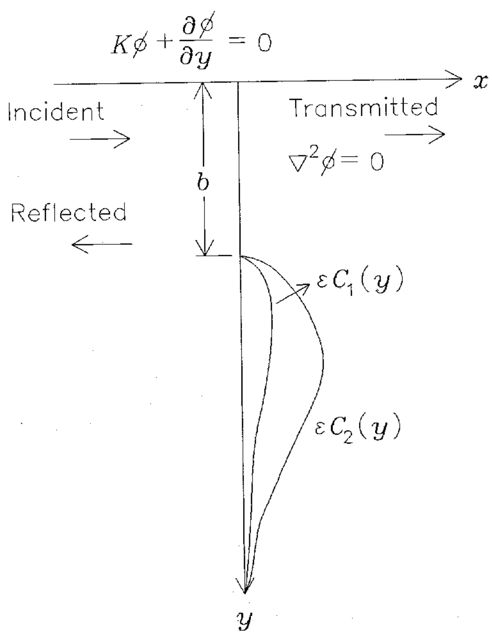

Figure 1b. Slender barrier $(b < y < \infty)$.

$$\phi \sim \begin{cases} T\exp(-Ky + iKx) & \text{as} \quad x \to \infty, \\ \exp(-Ky + iKx) + R\exp(-Ky - iKx) & \text{as} \quad x \to -\infty. \end{cases} \tag{6}$$

Here $K = \sigma^2/g$ is a real constant, g is the acceleration due to gravity, r denotes the distance from an edge, S denotes the two sides of the slender barriers and is given by

$$S = S_1 \bigcup S_2, S_j\colon x = \varepsilon C_j(y)(j = 1, 2), y \in L. \tag{7}$$

Here, for the problem P_1, say, which corresponds to the case of a partially immersed fixed slender barrier $L = L_1 = \{y\colon 0 < y < a\}$, and for the problem P_2, say, which corresponds to that of a completely submerged fixed slender barrier $L = L_2 = \{y\colon b < y < \infty\}, \varepsilon > 0$ is a small dimensionless quantity. The $C_j(y)$'s are bounded and continuous functions of y for the two problems with $C_j(0) = 0$ and $C_j(a) = 0$ for the problem P_1, and $C_j(b) = 0$ and $C_j(y) \to 0$ as $y \to \infty$ for the problem P_2. Further, $\frac{\partial}{\partial\nu}$ denotes the outward drawn normal derivative to S_j; T and R are the transmission and reflection coefficients (unknown), respectively, which are to be determined. It is assumed that the partially immersed slender barrier has a sharp edge at the lower end $(0, a)$ and the completely submerged barrier has a sharp edge at its top-most point $(0, b)$. The term $\exp(-Ky + iKx)$ represents a train of surface water waves incident upon the barriers from negative infinity, the y axis being taken vertically downwards into the fluid and passing through the upper and lower ends of the slender barriers. The situations under consideration are sketched in Figures 1(a) and 1(b).

In (7), the parameter ε is assumed to be very small and this gives a measure of the slenderness of the barriers. Thus, neglecting terms of the order $O(\varepsilon^2)$, we can approximate the boundary condition (3) as (*cf.* Shaw [1])

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x}(+0,y) &= \varepsilon\frac{\mathrm{d}}{\mathrm{d}y}\left\{C_1(y)\frac{\partial\phi}{\partial y}(+0,y)\right\} \\ \frac{\partial\phi}{\partial x}(-0,y) &= \varepsilon\frac{\mathrm{d}}{\mathrm{d}y}\left\{C_2(y)\frac{\partial\phi}{\partial y}(-0,y)\right\} \end{aligned}\quad \text{for}\quad y\in L,\right\} \tag{8}$$

where $L = L_1$ for P_1, and $L = L_2$ for P_2.

3. Solution by perturbational analysis

The form of the approximate boundary conditions (8) suggests that we may expand the unknown function $\phi(x,y)$, and the two unknown constants of physical interest R and T, in terms of the parameter ε, as (*cf.* MC [2])

$$\left.\begin{aligned} &\phi(x,y;\varepsilon) = \phi_0(x,y) + \varepsilon\phi_1(x,y) + O(\varepsilon^2), \\ &R = R_0 + \varepsilon R_1 + O(\varepsilon^2), \\ &T = T_0 + \varepsilon T_1 + O(\varepsilon^2). \end{aligned}\right\}. \tag{9}$$

Here we restrict our attention to the determination of the constants R_0, T_0, R_1 and T_1 only, as we are interested in evaluating only the corrections up to first order to the reflection and transmission coefficients. Using the expansions given by (9) in the basic partial differential equation (1), the BCs (2) and (8), the edge condition (4), and the infinity requirements (5) and (6), we obtain, after equating the coefficients of like powers of ε^0 and ε^1, from both sides of the results derived thus, that the functions ϕ_0 and ϕ_1 must be the solution of the following two independent BVPs:

BVP-I: The problem is to determine the function $\phi_0(x,y)$ satisfying

$$\nabla^2\phi_0 = 0 \quad \text{in} \quad y > 0,$$

$$K\phi_0 + \frac{\partial\phi_0}{\partial y} = 0 \quad \text{on} \quad y = 0,$$

$$\frac{\partial\phi_0}{\partial x} = 0 \quad \text{on} \quad x = 0, \quad y\in L_1 \quad \text{for} \quad P_1, \quad \text{and} \quad y \in L_2 \quad \text{for} \quad P_2,$$

$r^{1/2}\nabla\phi_0$ is bounded as $r \to 0$, where r is the distance from the sharp edges of the barriers,

$$\phi_0, \nabla\phi_0 \to 0 \quad \text{as} \quad y \to \infty,$$

$$\phi_0 \sim \begin{cases} T_0\exp(-Ky + iKx) & \text{as} \quad x \to \infty, \\ \exp(-Ky + iKx) + R_0\exp(-Ky - iKx) & \text{as} \quad x \to -\infty. \end{cases}$$

BVP-II: The problem is to determine the function $\phi_1(x,y)$ satisfying

$$\nabla^2\phi_1 = 0 \quad \text{in} \quad y > 0,$$

$$K\phi_1 + \frac{\partial \phi_1}{\partial y} = 0 \quad \text{on} \quad y = 0,$$

$$\left.\begin{aligned}\frac{\partial \phi_1}{\partial x}(+0,y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left\{C_1(y)\frac{\partial \phi_0}{\partial y}(+0,y)\right\}\\ \frac{\partial \phi_1}{\partial x}(-0,y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left\{C_2(y)\frac{\partial \phi_0}{\partial y}(-0,y)\right\}\end{aligned}\right\} \begin{aligned}&y \in L_1 \quad \text{for} \quad P_1 \quad \text{and}\\ &y \in L_2 \quad \text{for} \quad P_2.\end{aligned} \tag{10}$$

$r^{1/2}\nabla\phi_1$ bounded as $r \to 0$, where r is defined in BVP-I,

$$\phi_1, \nabla\phi_1 \to 0 \quad \text{as} \quad y \to \infty,$$

$$\phi_1 \sim \begin{cases} T_1 \exp(-Ky + iKx) & \text{as} \quad x \to \infty,\\ R_1 \exp(-Ky - iKx) & \text{as} \quad x \to -\infty.\end{cases}$$

Of the two problems, BVP-I and II, for the functions ϕ_0 and ϕ_1, the complete solution to the BVP-I is well known (*cf*. Dean [3], Ursell [4], Goswami [11], Mandal and Kundu [12]), and is given by

$$\left.\begin{aligned}\phi_0^{(1)}(x,y) &= \exp(-Ky + iKx) + R_0^{(1)} \exp(-Ky - iKx)\\ &\quad + \frac{1}{\Delta_1}\int_0^\infty \frac{J_1(ka)(k\cos ky - K\sin ky)\exp(kx)}{k^2+K^2}\,\mathrm{d}k, \quad (x<0),\\ \phi_0^{(1)}(x,y) &= T_0^{(1)} \exp(-Ky + iKx)\\ &\quad - \frac{1}{\Delta_1}\int_0^\infty \frac{J_1(ka)(k\cos ky - K\sin ky)\exp(-kx)}{k^2+K^2}\,\mathrm{d}k, \quad (x>0),\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}\phi_0^{(2)}(x,y) &= \exp(-Ky + iKx) + R_0^{(2)} \exp(-Ky - iKx)\\ &\quad - \frac{1}{\Delta_2}\int_0^\infty \frac{J_0(kb)(k\cos ky - K\sin ky)\exp(kx)}{k^2+K^2}\,\mathrm{d}k, \quad (x<0),\\ \phi_0^{(2)}(x,y) &= T_0^{(2)} \exp(-Ky + iKx)\\ &\quad + \frac{1}{\Delta_2}\int_0^\infty \frac{J_0(kb)(k\cos ky - K\sin ky)\exp(-kx)}{k^2+K^2}\,\mathrm{d}k, \quad (x>0),\end{aligned}\right\} \tag{12}$$

where

$$R_0^{(1)} = \pi I_1(Ka)/\Delta_1, \quad T_0^{(1)} = \mathrm{i}K_1(Ka)/\Delta_1, \quad \Delta_1 = \pi I_1(Ka) + \mathrm{i}K_1(Ka),$$

$$R_0^{(2)} = K_0(Kb)/\Delta_2, \quad T_0^{(2)} = \mathrm{i}\pi I_0(Kb)/\Delta_2, \quad \Delta_2 = K_0(Kb) + \mathrm{i}\pi I_0(Kb),$$

where $J_0, J_1, I_0, I_1, K_0, K_1$ are the standard Bessel functions, and $\phi_0^{(1)}, R_0^{(1)}, T_0^{(1)}$ represents the corresponding values of ϕ_0, R_0, T_0 for the problem P_1, and $\phi_0^{(2)}, R_0^{(2)}, T_0^{(2)}$ those for P_2.

Although the explicit solutions to the BVP-I, for the function ϕ_0, corresponding to the problems P_1 and P_2 are known, it is not easy to determine completely the function ϕ_1 which solves BVP-II corresponding to P_1 and P_2. On the other hand, the constants $R_1^{(1)}, T_1^{(1)}$

and $R_1^{(2)}, T_1^{(2)}$ corresponding to the problems P_1 and P_2, respectively, can be determined completely, without solving the problems, in the following manner:

For the problem P_1: To determine $R_1^{(1)}$ and $T_1^{(1)}$, we apply GT to the harmonic functions $\phi_0^{(1)}(x,y), \phi_1^{(1)}(x,y)$ and $\phi_0^{(1)}(-x,y), \phi_1^{(1)}(x,y)$, respectively, in the region bounded by the lines

$$\begin{aligned}
&0 < x \leqslant X, \quad y = 0; \quad x = 0+, \quad 0 \leqslant y \leqslant a;\\
&x = 0-, \quad 0 \leqslant y \leqslant a; \quad -X \leqslant x \leqslant 0, \quad y = 0;\\
&x = -X, \quad 0 \leqslant y \leqslant Y; \quad -X \leqslant x \leqslant X, \quad y = Y;\\
&x = X, \quad 0 \leqslant y \leqslant Y; \quad \text{with} \quad X, Y > 0
\end{aligned}$$

and a circle of small radius δ, having its centre at $(0,a)$, and allow both $X, Y \to \infty$, and $\delta \to 0$. Thus, using arguments similar to those in Evans [7] and the BCs (10) for the problem P_1, we find

$$\left.\begin{aligned}
R_1^{(1)} &= \mathrm{i}\int_0^a \left[C_1(y)\left\{\frac{\partial\phi_0^{(1)}}{\partial y}(+0,y)\right\}^2 - C_2(y)\left\{\frac{\partial\phi_0^{(1)}}{\partial y}(-0,y)\right\}^2\right] \mathrm{d}y,\\
T_1^{(1)} &= \mathrm{i}\int_0^a \{C_1(y) - C_2(y)\}\frac{\partial\phi_0^{(1)}}{\partial y}(+0,y)\frac{\partial\phi_0^{(1)}}{\partial y}(-0,y)\,\mathrm{d}y
\end{aligned}\right\} \tag{13}$$

in which the relations $C_j(0) = 0, C_j(a) = 0$ for $j = 1, 2$ have been used. Now, to derive analytical expressions for $R_1^{(1)}$ and $T_1^{(1)}$, we use the results for $\phi_0^{(1)}(\pm 0, y)$, which can be deduced by using (11), and is given by (*cf.* MC [2])

$$\phi_0^{(1)}(\pm 0, y) = \begin{cases} \exp(-Ky) \pm \dfrac{\exp(-Ky)}{\Delta_1 a} g(y) & \text{for} \quad y < a,\\ \exp(-Ky) & \text{for} \quad y \geqslant a, \end{cases}$$

$$\text{with} \quad g(y) = \int_a^y f(t)\exp(Kt)\mathrm{d}t, \quad \text{and} \quad f(t) = \frac{t}{(a^2 - t^2)^{1/2}}.$$

We can utilise the appropriate expressions of $\phi_0^{(1)}(\pm 0, y)$ in (13) to derive the general form of the analytical expressions for $R_1^{(1)}$ and $T_1^{(1)}$ which are given by

$$\begin{aligned}
R_1^{(1)} = \mathrm{i}\int_0^a \Bigg[&\{C_1(y) - C_2(y)\}\Bigg\{K^2\exp(-2Ky) + \frac{1}{\Delta_1^2 a^2} q(y)\\
&+ \frac{K^2}{\Delta_1^2 a^2}\exp(-2Ky)p(y) - \frac{2K}{\Delta_1^2 a^2}\exp(-Ky)f(y)g(y)\Bigg\}\\
&+ \frac{2K}{\Delta_1 a}\{C_1(y) + C_2(y)\}\{K\exp(-2Ky)g(y) - \exp(-Ky).f(y)\}\Bigg]\,\mathrm{d}y,
\end{aligned} \tag{14}$$

$$T_1^{(1)} = \mathrm{i}\int_0^a \{C_1(y) - C_2(y)\}\left[K^2\exp(-2Ky) - \frac{1}{\Delta_1^2 a^2}\{K\exp(-Ky)g(y) - f(y)\}^2\right]\mathrm{d}y, \tag{15}$$

where $p(y) = \{g(y)\}^2, q(y) = \{f(y)\}^2$.

Expressions (14) and (15) are the first-order corrections to the reflection and transmission coefficients for a surface water-wave train incident upon a fixed partially immersed slender barrier.

For the problem P_2: For this problem, to find the first-order corrections to the reflection and transmission coefficients, we again utilize Evans's [7] idea involving the application of GT to the harmonic functions $\phi_0^{(2)}(x,y), \phi_1^{(2)}(x,y)$ and $\phi_0^{(2)}(-x,y), \phi_1^{(2)}(x,y)$, respectively, in the region bounded by the lines

$$-X \leqslant x \leqslant X, \quad y = 0; \quad x = -X, \quad 0 \leqslant y \leqslant Y;$$

$$-X \leqslant x < 0, \quad y = Y; \quad x = 0-, \quad b \leqslant y \leqslant Y;$$

$$x = 0+, \quad b \leqslant y \leqslant Y; \quad y = Y, \quad 0 < x \leqslant X;$$

$$x = X, \quad 0 \leqslant y \leqslant Y; \quad \text{with} \quad X, Y > 0,$$

and a circle of small radius δ with its centre at $(0, b)$ and ultimately making X, Y tend to infinity and $\delta \to 0$, we obtain

$$\left.\begin{aligned} R_1^{(2)} &= \mathrm{i}\int_b^\infty \left[C_1(y)\left\{\frac{\partial\phi_0^{(2)}}{\partial y}(+0,y)\right\}^2 - C_2(y)\left\{\frac{\partial\phi_0^{(2)}}{\partial y}(-0,y)\right\}^2\right]\mathrm{d}y, \\ T_1^{(2)} &= \mathrm{i}\int_b^\infty \{C_1(y) - C_2(y)\}\frac{\partial\phi_0^{(2)}}{\partial y}(+0,y)\frac{\partial\phi_0^{(2)}}{\partial y}(-0,y)\,\mathrm{d}y. \end{aligned}\right\} \tag{16}$$

To derive (16), the BCs (10) for the problem P_2, and the conditions $C_j(b) = 0, C_j(y) \to 0$ as $y \to \infty$ for $j = 1, 2$, have been used. Now, we can use (12) to derive the expressions for $\phi_0^{(2)}(\pm 0, y)$, which are given by (*cf.* MC [2])

$$\phi_0^{(2)}(\pm 0, y) = \begin{cases} \exp(-Ky) & \text{for} \quad 0 < y \leqslant b \\ \exp(-Ky) \mp \dfrac{\exp(-Ky)}{\Delta^2}G(y) & \text{for} \quad y > b, \end{cases}$$

where $G(y) = \int_b^y F(t)\exp(Kt)\,\mathrm{d}t$ with $F(y) = (y^2 - b^2)^{-1/2}$. Using the expressions for $\phi_0^{(2)}(\pm 0, y)$ in (16), we obtain the analytical expressions for $R_1^{(2)}$ and $T_1^{(2)}$, which are given by

$$R_1^{(2)} = \mathrm{i}\int_b^\infty \left[\{C_1(y) - C_2(y)\}\left\{K^2\exp(-2Ky) + \frac{1}{\Delta_2^2}Q(y)\right.\right.$$

$$+\frac{K^2}{\Delta_2^2}\exp(-2Ky)P(y) - \frac{2K}{\Delta_2^2}\exp(-Ky)F(y)G(y)\Bigg\}$$

$$-\frac{2K}{\Delta_2}\{C_1(y)+C_2(y)\}\{K\exp(-2Ky)G(y) - \exp(-Ky)F(y)\}\Bigg]\,\mathrm{d}y, \tag{17}$$

$$T_1^{(2)} = \mathrm{i}\int_b^\infty \{C_1(y)-C_2(y)\}\Bigg[K^2\exp(-2Ky) - \frac{1}{\Delta_2^2}\{K\exp(-Ky).G(y)$$

$$-F(y)\}^2\Bigg]\,\mathrm{d}y, \tag{18}$$

with $P(y)=\{G(y)\}^2, Q(y)=\{F(y)\}^2$.

The Equations (17) and (18) provide the required expressions, for the first-order corrections to the reflection and transmission coefficients, for a surface water-wave train incident upon a fixed submerged slender barrier in deep water.

4. Special cases

(i) *Nearly vertical barrier*: If $C_1(y) = C_2(y) = C(y)$, then the slender-barrier problem reduces to that involving a nearly vertical barrier. Geometries are sketched in Figures 2(a) and 2(b). Thus, putting $C_1(y)=C_2(y)=C(y)$, we find for the problem P_1

$$R_1^{(1)} = \frac{4\mathrm{i}K}{\Delta_1 a}\int_0^a C(y)\{K\exp(-2Ky)g(y) - \exp(-Ky)f(y)\}\,\mathrm{d}y, \tag{19}$$

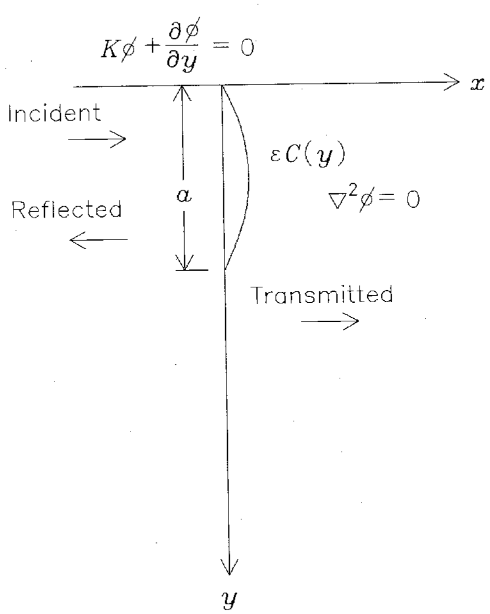

Figure 2a. Nearly vertical barrier $(0 < y < a)$.

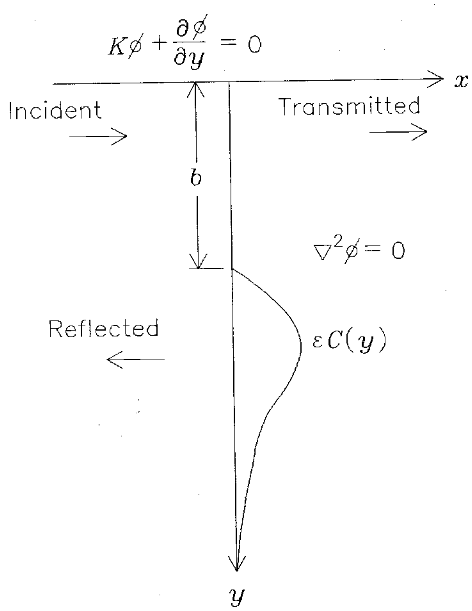

Figure 2b. Nearly vertical barrier $(b < y < \infty)$.

$$T_1^{(1)} = 0, \tag{20}$$

and for the problem P_2

$$R_1^{(2)} = -\frac{4\mathrm{i}K}{\Delta_2}\int_b^\infty C(y)\{K\exp(-2Ky)G(y) - \exp(-Ky)F(y)\}\,\mathrm{d}y, \tag{21}$$

$$T_1^{(2)} = 0. \tag{22}$$

Expressions (19) and (21) can be written, after some elementary manipulation, as

$$\begin{aligned}\frac{\mathrm{i}R_1^{(1)}}{4K}\Delta_1 a \;&=\; -K\int_0^a C(y)\exp(-2Ky)\left\{\int_a^y \frac{t\exp(Kt)}{(a^2-t^2)^{1/2}}\,\mathrm{d}t\right\}\mathrm{d}y\\ &\quad + \int_0^a C(y)\frac{y\exp(-Ky)}{(a^2-y^2)^{1/2}}\,\mathrm{d}y,\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\mathrm{i}R_1^{(2)}}{4K}\Delta_2 \;&=\; K\int_b^\infty C(y)\exp(-2Ky)\left\{\int_b^y \frac{\exp(Kt)}{(t^2-b^2)^{1/2}}\,\mathrm{d}t\right\}\mathrm{d}y\\ &\quad - \int_b^\infty C(y)\frac{\exp(-Ky)}{(y^2-b^2)^{1/2}}\,\mathrm{d}y.\end{aligned} \tag{24}$$

The expressions given by (23), (20), (24) and (22) are in complete agreement with those obtained by MC [2] in connection with the corresponding nearly vertical barrier problems.

(ii) *Vertically symmetric slender barrier*: When $C_1(y) = -C_2(y)$, the slender barrier becomes symmetric about the vertical (geometries are described in Figures 3(a) and 3(b)), and thus, assuming $C_1(y) = -C_2(y) = C(y)$, we find for the problem P_1

$$R_1^{(1)} \;=\; 2\mathrm{i}\int_0^a C(y)\left[K^2\exp(-2Ky) + \frac{1}{\Delta_1^2 a^2}\{K\exp(-Ky)g(y) - f(y)\}^2\right]\mathrm{d}y, \tag{25}$$

$$T_1^{(1)} \;=\; 2\mathrm{i}\int_0^a C(y)\left[K^2\exp(-2Ky) - \frac{1}{\Delta_1^2 a^2}\{K\exp(-Ky)g(y) - f(y)\}^2\right]\mathrm{d}y, \tag{26}$$

and for the problem P_2

$$R_1^{(2)} \;=\; 2\mathrm{i}\int_b^\infty C(y)\left[K^2\exp(-2Ky) + \frac{1}{\Delta_2^2}\{K\exp(-Ky)G(y) - F(y)\}^2\right]\mathrm{d}y, \tag{27}$$

$$T_1^{(2)} \;=\; 2\mathrm{i}\int_b^\infty C(y)\left[K^2\exp(-2Ky) - \frac{1}{\Delta_2^2}\{K\exp(-Ky)G(y) - F(y)\}^2\right]\mathrm{d}y. \tag{28}$$

The expressions (25)–(28) can be written, finally, in the form

$$R_1^{(1)} = A + B, \tag{29}$$

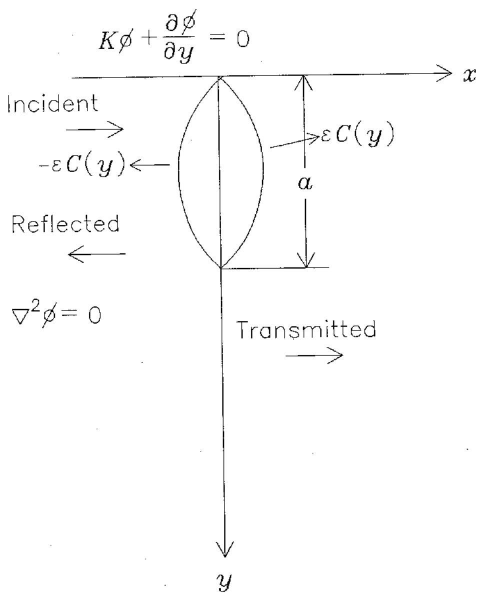

Figure 3a. Symmetric slender barrier $(0 < y < a)$.

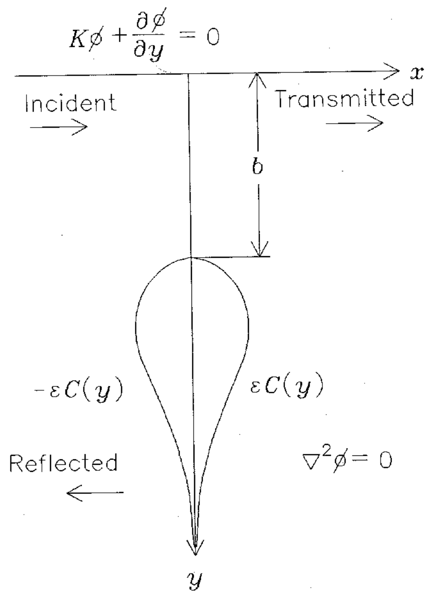

Figure 3b. Symmetric slender barrier $(b < y < \infty)$.

$$T_1^{(1)} = A - B, \tag{30}$$

$$R_1^{(2)} = C + D, \tag{31}$$

$$T_1^{(2)} = C - D, \tag{32}$$

where

$$A = 2\mathrm{i}K^2 \int_0^a C(y)\, \exp(-2Ky)\, \mathrm{d}y,$$

$$B = \frac{2\mathrm{i}}{\Delta_1^2 a^2} \int_0^a C(y) \left[K \exp(-Ky) \int_a^y \frac{t \exp(Kt)}{(a^2 - t^2)^{1/2}}\, \mathrm{d}t - \frac{y}{(a^2 - y^2)^{1/2}} \right]^2 \mathrm{d}y,$$

$$C = 2\mathrm{i}K^2 \int_b^\infty C(y)\, \exp(-2Ky)\, \mathrm{d}y,$$

$$D = \frac{2\mathrm{i}}{\Delta_2^2} \int_b^\infty C(y) \left[K \exp(-Ky) \int_b^y \frac{\exp(Kt)}{(t^2 - b^2)^{1/2}}\, \mathrm{d}t - \frac{1}{(y^2 - b^2)^{1/2}} \right]^2 \mathrm{d}y.$$

Equations (29)–(32) are the first-order corrections to the reflection and transmission coefficients for a surface water-wave train incident upon a fixed vertically symmetric slender barrier, which is either partially immersed or completely submerged. It should be noted here that, although the problem of diffraction of water waves by a vertically symmetric slender barrier submerged up to a finite depth below the FS has been formulated, briefly, by Shaw [1], wherein he employed the integral equation method, the first-order corrections to the reflection and transmission coefficients, which are the quantities of physical interest, are not given there.

5. Numerical results

The reflection and transmission coefficients up to $O(\varepsilon)$ are now evaluated numerically to support the theory presented in this paper. More specifically, we compute the reflection coefficient $|R|(=|R_0+\varepsilon R_1|)$ and the transmission coefficient $|T|(=|T_0+\varepsilon T_1|)$ here, correct up to seven decimal places, for different values of the various parameters, assuming some typical forms of $C_1(y)$ and $C_2(y)$, for both the problems P_1 and P_2.

In Figure 4, $|R|$ and $|T|$ are plotted graphically against different values of $Ka = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1{\cdot}0, 1{\cdot}4, 1{\cdot}8$ (as considered by Evans and Morris [13]) for $\varepsilon = 0{\cdot}001$ and $a = 5$ units of length, assuming $C_1(y) = y(a-y)/a$ and $C_2(y) = a\sin(\pi y/a), 0 < y < a$. From Figure 4, it is observed that $|R|$ first increases rapidly and then increases asymptotically to unity with the increase of Ka from 1·4; however, $|T|$ first decreases slowly, then rapidly and finally asymptotically to zero with the increase of Ka from 1·6. These results are expected, since, when Ka becomes large, the wave trains are confined near the FS and are almost reflected totally by the slender barrier.

For the problem P_2, assuming the two sides of the barrier as $C_1(y) = (y-b)\exp(-wy)$ and $C_2(y) = C_0 b(y-b)/y^2, b < y < \infty$, we have plotted $|R|$ and $|T|$ graphically for different values of $Kb = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1{\cdot}0, 1{\cdot}4, 1{\cdot}8$ in Figure 5, for $\varepsilon = 0{\cdot}001, w = 5{\cdot}0, C_0 = 0{\cdot}5b$ and $b = 1{\cdot}0$ unit of length. It is observed from Figure 5 that $|R|$ first decreases rapidly, and then decreases asymptotically to zero with the increase of Kb from 1·6; however, $|T|$ first increases rapidly, then increases asymptotically to unity with the increase of Kb from 0·6. These are plausible results, since, for large wave numbers, the waves are confined within a thin layer near the FS and almost all the wave energy is transmitted above the slender barrier submerged in water.

We have also calculated $|R|$ and $|T|$ numerically for various values of the different parameters, assuming $\varepsilon = 0{\cdot}005$. The results are shown in Figure 6 for the problem P_1,

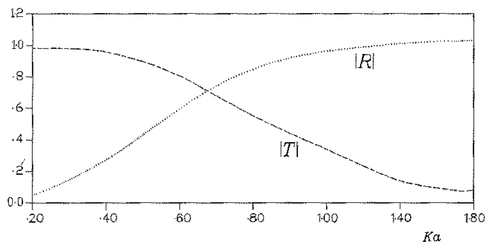

Figure 4. Reflection and transmission coefficients vs Ka when $0 < y < a$, $(\varepsilon = 0{\cdot}001)$.

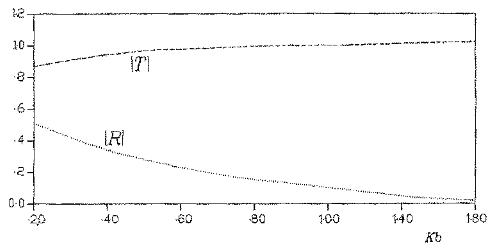

Figure 5. Reflection and transmission coefficients vs Kb when $b < y < \infty$, $(\varepsilon = 0{\cdot}001)$.

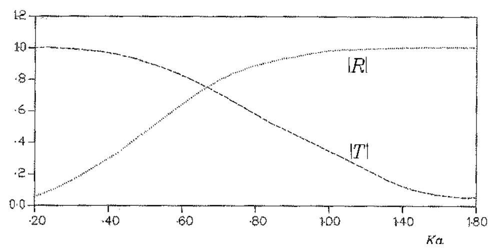

Figure 6. Reflection and transmission coefficients vs Ka when $0 < y < 0$, $(\varepsilon = 0{\cdot}005)$.

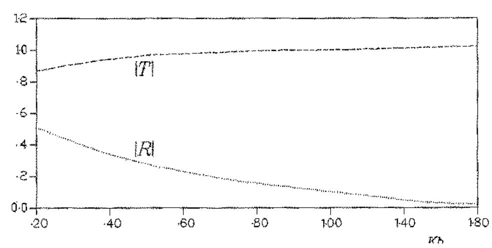

Figure 7. Reflection and transmission coefficients vs Kb when $b < y < \infty$, $(\varepsilon = 0{\cdot}005)$.

and in Figure 7 for the problem P_2. From the Figures 6 and 7 a similar qualitative nature, as already discussed in case of the Figures 4 and 5, is observed.

Figures 6 and 7 for $\varepsilon = 0{\cdot}005$ are drawn separately, since the values of $|R|$ and $|T|$ for $\varepsilon = 0{\cdot}001$ and $\varepsilon = 0{\cdot}005$ are so close together that any distinction between them is not possible to detect had we drawn them in Figures 4 and 5.

In Table 1 we have tabulated $|R|(= |R_0 + \varepsilon R_1^{(1)}|)$ for the nearly vertical barrier, as well as for the symmetric slender barrier, assuming $C(y) = y(a-y)/a, 0 < y < a$.

Table 1.

Ka	$\varepsilon = 0{\cdot}001$		$\varepsilon = 0{\cdot}005$	
	$\|R\|$(NVB)	$\|R\|$(SSB)	$\|R\|$(NVB)	$\|R\|$(SSB)
0·2	0·0660024	0·0660058	0·0660037	0·0660194
0·4	0·2816058	0·2816170	0·2816099	0·2816619
0·6	0·6033109	0·6033319	0·6033201	0·6034164
0·8	0·8447172	0·8447488	0·8447387	0·8448754
1·0	0·9470236	0·9470664	0·9470592	0·9472377
1·4	0·9934299	0·9934900	0·9934708	0·9937316
1·8	0·9990353	0·9991121	0·9990996	0·9994219

Table 2.

Ka	$\varepsilon = 0{\cdot}001$		$\varepsilon = 0{\cdot}005$	
	$\|R\|$(NVB)	$\|R\|$(SSB)	$\|R\|$(NVB)	$\|R\|$(SSB)
0·2	0·4840007	0·4840044	0·4840029	0·4840188
0·4	0·3230039	0·3230156	0·3230141	0·3230636
0·6	0·2210143	0·2210364	0·2210297	0·2211269
0·8	0·1520612	0·1520943	0·1520786	0·1522293
1·0	0·1050763	0·1051210	0·1051072	0·1053064
1·4	0·0499896	0·0500533	0·0500207	0·0503075
1·8	0·0233993	0·0234802	0·0234401	0·0238175

It is observed, from Table 1, that $|R|$ for nearly vertical barrier, as well as for a symmetric slender barrier, increases with increasing Ka, first rapidly and then approaches unity with the increase of Ka above 1·4.

For the submerged nearly vertical barrier, as well as for the submerged symmetric slender barrier, we have calculated $|R|(= |R_0 + \varepsilon R_1^{(2)}|)$, assuming $C(y) = (y-b)\exp(-wy), b < y < \infty$. The results have been tabulated in Table 2.

From Table 2 we observe that $|R|$, for the nearly vertical barrier, as well as for the symmetric slender barrier, first decreases rapidly and then approaches zero with the increase of Kb above 1·4. It is also observed, from the Tables 1 and 2, that the various values of $|R|$ for nearly vertical barriers in most cases differ from those for symmetric slender barriers in the fourth or fifth decimal place. As terms of $O(\varepsilon^2)$ are neglected throughout the analysis, this indicates that the influence of ε is not of much significance for these types of curved barrier. From the tables it can be seen that more wave energy is reflected by the symmetric slender barrier than the nearly vertical barrier. In fact, it is found that among these three types of barrier, *i.e.* slender barrier, nearly vertical barrier and symmetric slender barrier, maximum wave energy is reflected by the symmetric slender barrier and minimum wave energy is reflected by the slender barrier.

6. Discussion

A formal mathematical derivation, to find the analytical expressions, for the first-order corrections to the reflection and transmission coefficients R_1 and T_1, respectively, for a surface water wave train incident upon a fixed slender barrier, has been described in this paper. The barrier is assumed to be either partially immersed or completely submerged in deep water. A simple and straightforward perturbation technique devised by MC [2] is used to find R_1 and T_1. Analytical expressions for R_1 and T_1 (for both problems), over the vertical barrier results, have been obtained here in terms of non-trivial integrals involving the shape functions describing two sides of the slender barrier. Assuming the explicit forms of $C_1(y)$ and $C_2(y)$, we also calculated these non-trivial integrals numerically. As a special case, when two sides of the slender barrier becomes identical, the slender barrier reduces to a nearly vertical barrier and is observed to produce results involving corrections to the reflection and transmission coefficients which are in complete agreement with those found by Shaw [1] and MC [2]. These corrections were also derived for a vertically symmetric slender barrier which is either partially immersed or completely submerged.

Thus, in this paper, the problem of the diffraction of water waves by a slender barrier has been studied for two important cases, which to the best of the author's knowledge have not been addressed before, except for some passing remarks. It may, however, be mentioned that Shaw [1] stated the problem of the diffraction of water waves by a vertically symmetric slender barrier (which has been deduced, in this paper, as a special case of the slender barrier problem – see section 4(ii)), by employing a method based on an integral-equation formulation. But he has not touched upon solutions for the reflection and transmission coefficients. Viewed in this light, results obtained in the present paper may be regarded as the first-order corrections for the reflection and transmission coefficients associated with the slender-barrier problem. To the author, this is something new and has not been obtained before, even for vertically symmetric slender barriers. Determination of R_1 and T_1 is, therefore, one of the motives behind taking up the present investigation.

The reason for choosing a perturbation technique is that it provides an extremely elegant approach (although old-fashioned) for diffraction problems in water-wave theory which, when applied judiciously, produces the desired results fairly easily and relatively quickly as compared to the integral-equation technique. It must be admitted that the approximations involved in this work are expressed in terms of non-trivial integrals whose evaluation must be done numerically. The point is that we may evaluate these non-trivial integrals numerically by applying any one of the standard methods for numerical integration such as the Gauss – quadrature formula, Simpson's one-third rule etc. Thus, the numerical integrations required here are much simpler than the procedures required in any direct numerical solution. It may be mentioned here that, while the numerical solution of the full problem is relatively straightforward for a barrier of substantial thickness, it is likely to be singular as the thickness tends to zero, *i.e.* for the thin-barrier case. An analogous situation exists in thin-wing theory where penel methods cannot be used for arbitrarily thin-wing sections, unless the number of panels is increased to impractical levels.

To check the accuracy of the approximations to the reflection and transmission coefficients analytically, known results for a nearly vertical barrier (both for partially immersed and completely submerged), given by Shaw [1] and MC [2], have been deduced from that for a slender barrier (see section 4(i)). It is interesting to note that, although the correction to the transmission coefficient T_1 vanishes for a nearly vertical barrier (*cf.* Shaw [1], MC [2], see also section 4(i)), it does not vanish for a slender barrier or for a vertically symmetric slender barrier. These interesting results have not been presented so far for slender barriers, nor for vertically symmetric slender barriers.

The approximate analytical approach described in this paper gives additional insight over numerical solutions and we may make further analytical progress:
(i) by choosing particular shape function for the curved barrier;
(ii) by determining R_2 and T_2, the-second order corrections to the reflection and transmission coefficients, respectively. To determine R_2 and T_2 by means of a perturbation technique, it is necessary to find $\phi_1(x, y)$ for $x > 0$, as well as for $x < 0$. The explicit form of $\phi_1(x, y)$,

the solution of the BVP-II, can be found by the procedure adopted by Vijaya-Bharathi and Chakrabarti [9] for the submerged nearly vertical barrier extending infinitely downwards or, by Mandal and Banerjea [10] for the partially immersed nearly vertical barrier.

7. Conclusion

The conclusions reached based on the present investigation are given as follows:

1. As compared to the integral-equation technique the perturbation technique employed here is a simple and straightforward technique to solve the diffraction problem.
2. The final important results derived in this paper are the first-order corrections to the reflection and transmission coefficients *i.e.* R_1 and T_1, for the slender barrier problem, which have not been derived so far.
3. Analytical expressions for R_1 and T_1 are derived here in terms of non-trivial integrals and evaluation of these integrals is much simpler than any direct numerical solution of the full problem would be.
4. As a check, the results obtained here are compared with the known results for a nearly vertical barrier given by Shaw [1] and MC [2].
5. Although T_1 vanishes for a nearly vertical barrier, it does not vanish for a slender barrier or for a vertically symmetric slender barrier.
6. Further analytical development of the problem studied here may be done.

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